IV. *A ſecond Letter from Mr.* Colin M^c Laurin, *Profeſſor of Mathematicks in the Univerſity of* Edinburgh *and F. R. S. to* Martin Folkes, *Eſq; concerning the Roots of Equations, with the Demonſtration of other Rules in Algebra; being the Continuation of the Letter publiſhed in the* Philoſophical Tranſactions, N° 394.

Edinburgh, April 19th, 1729.

SIR,

IN the Year 1725, I wrote to you that I had a Method of demonſtrating Sir *Iſaac Newton*'s Rule concerning the impoſſible Roots of Equations, deduced from this obvious Principle, that the Squares of the Differences of real Quantities muſt always be poſitive; and ſome time after, I ſent you the firſt Principles of that Method, which were publiſhed in the *Philoſophical Tranſactions* for the Month of *May*, 1726. The Deſign I have for ſome Time had of publiſhing a Treatiſe of Algebra, where I propoſed to treat this and ſeveral other Subjects in a new Manner, made me think it unneceſſary to ſend you the remaining Part of that Paper. But ſome Reaſons have now determined me to ſend you with the Continuation of my former Method, a ſhort Account of two other Methods in which I have treated the ſame Subject, and ſome Obſervations on Equations that I take to be new, and which will, perhaps, be more acceptable to you than what relates to the imaginary Roots themſelves. Beſides Sir *Iſaac Newton*'s Rule, there ariſes from the following general

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ral Propoſitions, a great Variety of new Rules, different from his, and from any other hitherto publiſhed, for diſcovering when an Equation has imaginary Roots. I ſhall particularly explain one that is more uſeful for that Purpoſe, than any that have been hitherto publiſhed.

Suppoſe there is an Equation of (n) Dimenſions of this Form,

$$x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + Fx^{n-6} - Gx^{n-7} + Hx^{n-8} - Ix^{n-9} + Kx^{n-10} \text{ \&c.} = 0.$$

And that the Roots of this Equation are, $a, b, c, d, e, f, g, h, i, k, l$, &c. then ſhall $A = a + b + c + d + e + f$ &c. and therefore I call a, b, c, d, e, f, &c. *Parts* or *Terms* of the Coefficient A. For the ſame Reaſon I call $ab, ac, ad, ae, bc, bd, cd$, &c. Parts or Terms of the Coefficient B; abc, abd, abe, acd, bcd, &c. Parts or Terms of C; $abcd, abce, abcf$, Parts or Terms of the Coefficient D, and ſo on. By the *Dimenſions* of any Coefficient; I mean the Number of Roots or Factors that are multiplied into each other in its Parts, which is always equal to the Number of Terms in the Equation that preceed that Coefficient. Thus A is a Coefficient of one Dimenſion, B of two, C of three, and ſo of the reſt. I call a Part or Term of a Coefficient C *ſimilar* to a Part or Term of any Coefficient G, when the Part of G involves all the Factors of the Part of C: Thus abc, $abcdefg$ are ſimilar Parts of C and G; after the ſame manner $abcd$, $abcdef$ are ſimilar Parts of D and F, the Part of F involving all the Factors of the Part of D. Thoſe I call *diſſimilar* Parts that involve no common Root or Factor: Thus abc, and $defgh$ are diſſimilar Parts of the Coefficients C and F. The Sum of all the Products that can be made by multiplying the Parts of any Coefficient C by all the ſimilar Parts of G, I expreſs by C′G′ placing a ſmall Line over each Coefficient: After the ſame manner D′F′ expreſſes the Sum of all the Products that can be made by multiplying the ſimilar Parts of D and F by each other; and C′×C′ expreſſes the Sum of the Squares of the Parts of the Coefficient C, but C′×C; expreſſes the Sum of the Products that can be made by multiplying any two Parts of C by one another. Theſe Expreſſions being underſtood, and the five Propoſitions in *Phil. Tranſ.* N° 394, being premiſed, next follows

PROP. VI.

If the Difference of the Dimenſions of any two Coefficients C *and* G *be called* (m) *then ſhall the Product of theſe Coefficients multiplied by one another be equal to*

$$C'G' + \overline{m+2} \times B'H' + \frac{m+3}{1} \times \frac{m+4}{2} A'I' + \frac{m+4}{1} \times \frac{m+5}{2} \times \frac{m+6}{3} \times I \times K.$$

Where B and H are the Coefficients adjacent to the Coefficients C and G, A and I the Coefficients adjacent to B and H, I and K the Coefficients adjacent to B and H.

It is known that $C = abc + abd + abe + abf + abg$, &c. and $G = abcdefg + abcdefh + abcdefi + bcdefgh$, &c. and it is manifeſt,

1. That in the Product CG each Term of C′G′ will ariſe once as $a^2b^2c^2defg$. But

2. Any Term of B′ H′ as $a^2b^2cdefgh$ may be the Product of abc, and $abdefgh$, or of abd and $abcefgh$, or of abe and $abcdfgh$, or of abf and $abcdegh$,

$abcdegh$, or of abg and $abcdefh$, or laſtly of abh and $abcdefg$; ſo that it may be the Product of any Term of C that involves with ab one of the Roots, c,d,e,f,g,h, multiplied by that Term of G, which involves ab and the other five; that is, it may ariſe in the Product C G as often as there are Roots in $a^2b^2cdefgh$ beſides a and b, or in general, as often as there are Units in the Difference of the Dimenſions of B and H, that is, $m+2$ times; becauſe m expreſſes the Difference of the Dimenſions of C and G, and conſequently in expreſſing the Value of C G the Coefficient of the ſecond Term B′H′ muſt be $m+2$.

3. Any Term of A I, as $a^2bcdefghi$, may be the Product of any Part of C that involves the Root a with any two of the reſt b,c,d,e,f,g,h,i (the Number of which is the Difference of the Dimenſions of A and I, which is in general equal to $m+4$) multiplied by the Part of G that involves a and the other ſix; and therefore $a^2bcdefghi$ or any other Term of A′I′ muſt ariſe as often as different Products of two Quantities can be taken from Quantities whoſe Number is $m+4$, that is $\overline{m+4}\times\frac{m+4-1}{2}$ times or $\frac{m+3}{1}\times\frac{m+4}{2}$ times; and conſequently in expreſſing the Value of C G the Coefficient of the third Term A′I′ muſt be $\frac{m+3}{1}\times\frac{m+4}{3}$.

4. Any Term of $1\times K$ as $abcdefghik$, may be the Product of any Part of C that involves three of its Factors, and of the Part of G that involves the reſt, and therefore may ariſe in the Product C G as often as different

Products of three Quantities can be taken out of Quantities whoſe Number is $m+6$ that is, $\overline{m+6}\times\frac{m+5}{2}\times\frac{m+4}{3}$ times, and therefore the Coefficient of the fourth Term in the Value of C G muſt be $\frac{m+4}{1}\times\frac{m+5}{2}\times\frac{m+6}{3}$.

In general, in expreſſing the Value of the Product of any two Coefficients C and G, if x expreſs the Order of any Term of this Value as $A'I'$, that is, the Number of Terms that precede it, the Coefficient of that Term muſt be $\frac{2x+m}{1}\times\frac{2x+m-1}{2}\times\frac{2x+m-2}{3}$ &c. taking as many Factors as there are Units in x.

COR. I. If it is required to find by this Propoſition the Square of any Coefficient E, then ſuppoſe $m=0$, the Difference of the Dimenſions of the Coefficients in this Caſe vaniſhing, and we ſhall have $E^2 = E'\times E' + 2D'F' + 3\times\frac{4}{2}\times C'G' + 4\times\frac{5}{2}\times\frac{6}{3}\times B'H'$ &c. $= E'\times E' + 2D'F' + 6C'G' + 20B'H' + 70A'I' + 252K$. Therefore if $E'\times E_{,}$ expreſs the Sum of the Products of any two parts of E multiplied by each other, we ſhall have $E^2 = E'\times E' + 2E'\times E_{,}$, and therefore $E'\times E_{,} = D'F' + 3C'G' + 10B'H' + 35A'I' + 126K$.

COR. II. It follows from this Propoſition that

$$E^2 = E' \times E' + 2D'F' + 6C'G' + 20B'H' + 70A'I' + 252K.$$
$$DF = D'F' + 4C'G' + 15B'H' + 56A'I' + 210K.$$
$$CG = C'G' + 6B'H' + 28A'I' + 120K$$
$$BH = B'H' + 8A'I' + 45K$$
$$AI = A'I' + 10K$$
$$K = K.$$

COR. III. It eaſily appears by comparing the Theorems given in the laſt Corollary, that

$$E'E' = E^2 - 2DF + 2CG - 2BH + 2AI - 2K.$$
$$D'F' = DF - 4CG + 9BH - 16AI + 25K$$
$$C'G' = CG - 6BH + 20AI - 50K$$
$$B'H' = BH - 8AI + 35K$$
$$A'I' = AI - 10K.$$

PROP. VII.

Let $l = n \times \frac{n-1}{2} \times \frac{n-2}{3}$ &c. *taking as many Factors as the Coefficient* E *has Dimenſions and* $\frac{l-1}{2l} \times E^2$ *ſhall always exceed* $DF - CG + BH - AI + K$ *when the Roots of the Equation are all real Quantities.*

For it is manifeſt that l expreſſes the Number of Parts or Terms in the Coefficient E, and it is plain from Propoſition V (*See Phil. Tranſ.* N° 394) that $\frac{l-1}{2l} \times E^2$ muſt always be greater than the Sum of the Products that can be made by multiplying any two of the Parts of E by each other, that is, than $E' \times E_{,}$; but $2E' \times E_{,} = E^2 - E'E' =$ (by the firſt Theorem in the laſt Corollary) $2DF - 2CG + 2BH - 2A'I + 2K$, and therefore ſince $\frac{l-1}{2l} \times E^2$ muſt always exceed $E'E_{,}$, it follows that $\frac{l-1}{2l} E^2$ muſt always be greater than $DF - CG + BH - AI + K$ when the Roots of the Equation are real Quantities.

SCHOL. In following my Method this was the firſt general Propoſition preſented it ſelf. For having firſt obſerved that if l expreſſes the Number of any Quantities, the Square of their Sum multiplied by $\frac{l-1}{2l}$ muſt always exceed the Sum of the Products made by multiplying any two of them by each other; and that the Exceſs was the Sum of the Squares of the Differences of the Quantities divided by $2l$, it was eaſy to ſee in the Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4}$ &c. $= 0$. Since B is the Sum of the Products of any two of the Parts of A, that if l expreſſes the Number of the Roots of the Equation, $\frac{l-1}{2l} \times A^2$ muſt always exceed B; and this is one Part of the 5th Propoſition. In the next Place, I compared the Sum of the Products of any two Parts of B with AC, and found that it was not equal to AC but to $AC - D$ from which I inferred, that if l expreſſes the

the Number of the Parts of B then $\frac{l-1}{2l}\times B^2$ muſt always exceed $AC-D$; and theſe eaſily ſuggeſted this general Propoſition.

PROP. VIII.

Let r *expreſs the Dimenſions of the Coefficient* C, *and* s *the Difference of the Dimenſions of the Coefficients* C *and* G, *then* B *and* H *being Coefficients adjacent to* C *and* G, $\overline{n-r-s}\times r\,C'G'$ *ſhall always be greater than* $\overline{s+1}\times\overline{s+2}\times B'H'$ *when the Roots of the Equation are all real Quantities affected with the ſame Sign.*

For taking the Differences of all thoſe Parts of the Coefficient C that are ſimilar in all their Factors but one, as abc, abh, abi, &c. and multiplying the Square of each Difference by ſuch Parts of the Coefficient D (which is of s Dimenſions) as are diſſimilar to both the Parts of C in that Difference, the Sum of all thoſe Squares thus multiplied, will conſiſt of Terms of $C'G'$ taken poſitively, and of Terms of $B'H'$ taken negatively. By multiplying in this manner $\overline{abc-abh}|^2+\overline{abc-abi}|^2+\overline{abc-abk}|^2$ &c. $+\overline{abc-ach}|^2+\overline{abc-aci}|^2+\overline{abc-ack}|^2$ &c. $+\overline{abc-bch}|^2+\overline{abc-bce}|^2+\overline{abc-bck}|^2$ &c. by $defg$ the Term of D, that is diſſimilar to all thoſe Parts of C, you will find that $a^2b^2c^2defg$ will ariſe in the Sum of the Products $r\times\overline{n-r-s}$ times: For thoſe Products may be alſo expreſſed thus $defga^2b^2\times\overline{\overline{c-h}|^2+\overline{c-i}|^2+\overline{c-k}|^2}$ &c. $+defga^2c^2\times\overline{\overline{b-h}|^2+\overline{b-i}|^2+\overline{b-k}|^2}$ &c. $+$ $defgb^2c^2$

$defgb^2c^2 \times \overline{\overline{a-h}|^2 + \overline{a-1}|^2 + \overline{a-k}|^2}$ &c. where the Number of the Differences $c-h$, $c-i$, $c-k$, &c. whoſe Squares are multiplied by $defga^2b^2$ is manifeſtly equal to the Number of the Roots of the Equation that do not enter $a^2b^2c^2defg$ or $abcdefg$, that is, to the Exceſs of the Number of the Roots of the Equation above the Dimenſions of $abcdefg$, a Term of G, that is to $n-r-s$. But in collecting all the ſaid Products, $\overline{n-r-s} \times a^2b^2c^2defg$ muſt ariſe as often as there are Units in r: Becauſe the Terms which are ſubtracted from abc may differ from it in the Root c, as abh, abi, abk, &c. or in the Root b, as ach, aci, ack, &c. or in the Root a as bch, bci, bck; that is, $\overline{n-r-s} \times a^2b^2c^2defg$ muſt ariſe as often as there are Dimenſions in abc, a Term of C, or as often in general as there are units in r, which expreſſes the Dimenſions of C: Therefore the Term $a^2b^2c^2defg$ will ariſe in the Sum of the above-mentioned Products $r \times \overline{n-r-s}$ times.

The Negative Part muſt conſiſt of the Terms of $B'H'$ doubled; each of which, as $2a^2b^2cdefg$ may ariſe as often as there can be Differences $c-d$, $c-e$, $c-f$, $c-g$, $d-e$, &c. aſſumed amongſt the Terms c,d,e,f,g whoſe Number is equal to $s+2$ that is, $\overline{s+2} \times \frac{s+1}{2}$ times; and therefore a^2b^2cdefg or any other Part of $B'H'$ muſt ariſe in the negative Part $\overline{s+1} \times \overline{s+2}$ times; and ſince the whole aggregate muſt be poſitive it follows $\overline{n-r-s} \times rC'G'$ muſt always exceed $\overline{s+1} \times \overline{s+2} \times B'H'$.

Cor. I. Suppoſe we are to compare $E'E'$ the Sum of the Squares of the Parts of E with $D'F'$ the Sum of the Products of the ſimilar Parts of D and F; in this Caſe s vaniſhes, and therefore $\overline{n-r}\times r\,E'E'$ muſt exceed $2\,D'F'$. Let $\overline{n-r}\times r=m$ and conſequently $\overline{n-r-1}\times\overline{r-1}=m-n+1$; $\overline{n-r-2}\times\overline{r-2}=m-2n+4$; $\overline{n-r-3}\times\overline{r-3}=m-3n+9$; $\overline{n-r-4}\times\overline{r-4}=m-4n+16$. Since it is plain that $\overline{n-r-q}\times\overline{r-q}=\overline{n-r}\times r-qn+q^2$. Then by this Propoſition, ſuppoſing

$$m\times E'E'-2D'F'=a'$$
$$\overline{m-n+1}\times D'F'-12C'G'=b'$$
$$\overline{m-2n+4}\times C'G'-30B'H'=c'$$
$$\overline{m-3n+9}\times B'H'-56A'I'=d'$$
$$\overline{m-4n+16}\times A'I'-90K'=e'$$

The Quantities a',b',c',d',e', muſt be always poſitive when the Roots of the Equation are real Quantities affected with the ſame Sign. The Coefficients prefixed to the negative Parts are the Numbers 2,12,30,56,90, whoſe Differences equally increaſe by the ſame Number 8.

Cor. II. Suppoſing as before, that $\overline{n-r}\times r=m$; and alſo that $m\times\overline{m-n+1}=m'$; $m'\times\overline{m-2n+4}=m''$; $m''\times\overline{m-3n+9}=m'''$ &c. it may be demonſtrated after the manner of this Propoſition, that if

$$mE'E'-2D'F'=a'$$
$$m'E'E'-2\times12C'G'=a''$$
$$m''E'E'-2\times12\times30B'H'=a'''$$
$$m'''E'E'-2\times12\times30\times56A'I'=a''''\ \text{\&c.}$$

Then ſhall a', a'', a''', &c. be always poſitive when the Roots are real Quantities, whether they be affected with the ſame, or with different Signs. The Negative Coefficients ariſe by multiplying thoſe in the preceeding Corollary, 2, 12, 30, 56, 90, by one another.

PROP. IX.

Let a', b', c', d', e', *and* m *expreſs the ſame Quantities as in the Corollaries of the laſt Propoſition, and* $mE^2 - \overline{m+n+1} \times DF = a' + b' + 2c' + 5d' + 14e'$.

For by Cor. ii Prop. vi.

$E^2 = E'E' + 2D'F' + 6C'G' + 20B'H' + 70A'I' + 252K$,

and by the ſame

$DF = \quad - \; - \; - \; D'F' + 4C'G' + 15B'H' + 56A'I' + 210K$;

therefore $mE^2 - \overline{m+n+1} \times DF = mE'E' + \overline{m-n-1} \times D'F' + \overline{m-2n-2} \times 2C'G' + \overline{m-3n-3} \times 5B'H' + \overline{m-4n-4} \times 14A'I' + \overline{m-5n-5} \times 42K =$ (by ſubſtituting ſucceſſively for $mE'E'$, $\overline{m-n+1} \times D'F'$, $\overline{m-2n+4} \times C'G'$, $\overline{m-3n+9} \times B'H'$, $\overline{m-4n+16} \times A'I'$ their Values deduced from the firſt Corollary of the laſt Propoſition) $= a' + b' + 2c' + 5d' + 14e'$, where the Coefficients prefixed to a', b', c', d', e', are the Differences of the Coefficients of $E'E'$, $D'F'$, $C'G'$, $B'H'$, $A'I'$ and K in the Values of E^2 and DF taken from Cor. ii. Prop. vi. being $1-0$, $2-1$, $6-4$, $20-15$, $70-56$ and $252-210$.

Cor. Since $m = \overline{n-r} \times r$ therefore $m+n+1 = \overline{n-r+1} \times \overline{r+1}$; and conſequently $\frac{r}{r+1} \times \frac{n-r}{n-r+1} \times E^2$ muſt always be greater than D F the Product of the Coefficients adjacent to E; and hence the Fractions are deduced, that in Sir *Iſaac Newton*'s Rule are placed over the Terms of the Equation, which multiplied by the Square of the Terms under them, muſt always exceed the Products of the adjacent Terms of the Equation, when the Roots are real Quantities: For it is manifeſt that the Fraction to be placed over the Term Ex^{n-r} according to that Rule is the Quotient of $\frac{n-r}{r+1}$ divided by $\frac{n-r+1}{r}$.

PROP. X.

The ſame Expreſſions being allowed as in the preceeding Propoſitions, it will be found in the ſame manner that as $mE^2 - \overline{m+n+1} \times DF = a'+b'+2c'+5d'+14e'$ ſo
$\overline{m-n+1} \times DF - \overline{m+2n+4} \times CG =$ - $b'+3c'+9d'+28e'$
$\overline{m-2n+4} \times CG - \overline{m+3n+9} \times BH =$ - - $c'+5d'+20e'$
$\overline{m-3n+9} \times BH - \overline{m+4n+16} \times AI =$ - - - - $d'+7e'$
$\overline{m-4n+16} \times AI - \overline{m+5n+25} \times K =$ - - - - - - e.

Theſe Theorems are eaſily deduced from the Theorems given in the ſecond Corollary of Prop. vi. and the firſt Corollary of the viiith Propoſition; and the Coefficients prefixed to a', b', c', d', e', are the Differences of the Coefficients of the correſponding Terms in the Values of E^2, D F, C G, B H, A I and K in Cor. ii. Prop. vi.

Cor. Hence the Products of any two Coefficients, as D F and A I may be compared together when the Sum of the Dimenſions of D and F is equal to the Sum of the Dimenſions of A and I. Let the Dimenſions of A and F be equal to s and m reſpectively, and let $p = \frac{n-s}{s+1} \times \frac{n-s-1}{s+2} \times \frac{n-s-2}{s+3}$ &c. taking as many Factors as there are Units in the Difference of the Dimenſions of D and A. Let $q = \frac{n-m}{m+1} \times \frac{n-m-1}{m+2} \times \frac{n-m-2}{m+3}$ &c. taking as many Factors as you took in the Value of p. Then ſhall $\frac{q}{p} \times$ DF always exceed AI when the Roots of the Equation are real Quantities affected with the ſame Sign; and this Rule obtains, though the Roots are affected with different Signs when the Coefficients D and F are equal.

PROP. XI.

The ſame Things being ſuppoſed as in the preceeding Propoſitions.

$$1.\ \left.\begin{aligned} &mE^2 - \overline{m+1} \times 2DF + \overline{m+4} \times 2CG - \overline{m+9} \times \\ &2BH + \overline{m+16} \times 2AI - \overline{m+25} \times 2K \ \text{- -} \end{aligned}\right\} = a'.$$

$$2.\ \left.\begin{aligned} &\overline{m-n+1} \times DF - \overline{m-n+4} \times 4CG + \overline{m-n+9} \times \\ &9BH - \overline{m-n+16} \times 16AI + \overline{m-n+25} \times 25K \end{aligned}\right\} = b'.$$

$$3.\ \left.\begin{aligned} &\overline{m-2n+4} \times CG - \overline{m-2n+9} \times 6BH + \overline{m-2n+16} \times \\ &20AI + \overline{m-2n+25} \times 50K \ \text{- - - - - -} \end{aligned}\right\} = c'.$$

$$4.\ \overline{m-3n+9} \times BH - \overline{m-3n+16} \times 8AI + \overline{m-3n+25} \times 35K = d'.$$

$$5.\ \text{- -}\ \overline{m-4n+16} \times AI - \overline{m-4n+25} \times 10K = e'.$$

Theſe Theorems follow eaſily from the third Corollary of the vi^th^ Propoſition. The firſt eaſily appears thus, $a' = mE'E' - 2D'F' =$ (by that Corollary)

$$mE^2 - 2mDF + 2mCG - 2mBH + 2mAI - 2mK.$$
$$-2DF + 8CG - 18BH + 32AI - 50K.$$

$$= mE^2 - \overline{m+1} \times 2DF + \overline{m+4} \times 2CG - \overline{m+9} \times 2BH + \overline{m+16} \times 2AI - \overline{m+25} \times 2K.$$

The other Theorems are deduced from the ſame Corollary compared with Cor. i. Prop. viii.

P R O P. XII.

The ſame Things being ſuppoſed as in the ſecond Corollary of the viii^th^ *Propoſition.*

1. $\left.\begin{matrix} mE^2 - \overline{m+1} \times 2DF + \overline{m+4} \times 2CG - \overline{m+9} \times \\ 2BH + \overline{m+16} \times 2AI - \overline{m+25} \times 2K \ \text{- - -} \end{matrix}\right\} = a'.$

2. $\left.\begin{matrix} m'E^2 - 2m'DF + \overline{m'-12} \times 2CG - \overline{m'-72} \times \\ 2BH + \overline{m'-240} \times 2AI - \overline{m'-600} \times 2K \ \text{- - -} \end{matrix}\right\} = a''.$

3. $\left.\begin{matrix} m''E^2 - 2m''DF + 2m''CG - \overline{m''+360} \times \\ 2BH + \overline{m'''+360 \times 8} \times 2AI - \overline{m''+360 \times 35} \times 2K \end{matrix}\right\} = a'''.$

4. $\left.\begin{matrix} m''' \times \overline{E^2 - 2DF + 2CG - 2BH} + \overline{m''-750 \times 28} \times \\ AI - \overline{m''''-7200 \times 28} \times 2K \ \text{- - - - - - -} \end{matrix}\right\} = a''''.$

&c.

Theſe Theorems follow from the third Corollary of the vi^th^ Propoſition compared with the ſecond Corollary of the eighth Propoſition. The firſt is the ſame with the firſt of the laſt Propoſition. The ſecond is demonſtrated by ſubſtituting in $m'E'E' - 24C'G' = a''$. The Values of $E'E'$ and $C'G'$ given in the third

Cor. of the vith Propoſition. The third is found by ſubſtituting in $m''E'E' - 720B'H' = a'''$ the Values of $E'E'$ and $B'H'$; and by a like Subſtitution theſe Theorems may be continued.

A General COROLLARY.

From theſe Propoſitions a great Variety of Rules may be deduced for diſcovering when an Equation has imaginary Roots. The Foundation of Sir *Iſaac Newton*'s Rule is demonſtrated in the ninth Propoſition, and its Corollary. The ſeventh Propoſition ſhews that if $\frac{l-1}{2l} \times E^2$ does not exceed $DF - CG + BH - AI + K$, ſome of the Roots of the Equation muſt be imaginary; and ſometimes this Rule will diſcover impoſſible Roots in an Equation, that do not appear by Sir *Iſaac Newton*'s Rule. Theſe are the only two Rules that have been hitherto publiſhed. But the Rules that ariſe from the Theorems in the eleventh and twelfth Propoſitions, are preferable to both; becauſe any imaginary Roots that can be diſcovered by the viith or ixth always appear from the xith and xiith Propoſitions; and impoſſible Roots will often be diſcovered by the xith and xiith Propoſitions in an Equation, that do not appear in that Equation when examined by the viith and ixth Propoſitions. The Advantage which the Rules deduced from the xith Propoſition, have above thoſe deduced from the preceeding Propoſitions, will be manifeſt by conſidering that in the xith Propoſition we have the Values of the Quantities a', b', c', d', e', ſeparately; whereas in the preceeding Propoſitions, we have only the Values of certain Aggregates of theſe Quantities joined with the ſame Signs. Now it is obvious that if theſe Quantities be ſeparately found poſitive, any ſuch Aggregates of them muſt be poſitive; but theſe Aggregates may be poſitive, and yet ſome of the Quantities a', b', c', d', e', themſelves may be found negative: From which it follows, that if the Roots of the Equation are all affected with the ſame Sign, and no impoſſible Roots appear by Propoſition xith, none will appear by the preceeding Propoſitions; but that ſome imaginary Roots may be diſcovered by Propoſition xith, when none appear in the Equation examined by the Propoſitions that preceed the xith. If ſome of the Roots of the Equation are poſitive, and ſome negative (which always eaſily appears by conſidering the Signs of the Terms of the Equation) then the xiith Propoſition will be in many Caſes more apt to diſcover imaginary Roots in an Equation than thoſe that preceed it.

The Rule that flows from the firſt Theorem of the xith Propoſition, obtains when the Roots of the Equation are affected with different Signs, as well as when they all have the ſame Sign, and it is this; Multiply the Number of the Terms in an Equation that preceeds any Term, as Ex^{n-r} by the Number of Terms that follow it in the ſame Equation, and call the Product m. Suppoſe that $+D$, $-C$, $+B$, $-A$, $+1$ are the Coefficients preceeding the Term Ex^{n-r}, and that $+F$, $-G$, $+H$, $-I$, $+K$ are the Coefficients that follow it; then if $\frac{1}{2}mE^2$ does not exceed $\overline{m+1}\times DF - \overline{m+4}\times CG + \overline{m+9}\times BH - \overline{m+16}\times AI + \overline{m+25}\times K$ the Equation muſt have ſome imaginary Roots; where the Coefficients $m+1$, $m+4$, $m+9$, &

&c. are found by adding to m the Squares of the Numbers 1, 2, 3, 4, &c. which ſhew the Diſtances of the Coefficients to which they are prefixed, from the Coefficient E. The ſecond Theorem of the xiith Propoſition ſhews, that if $\frac{1}{2}m' E^2$ does not exceed $m' DF - \overline{m' - 12} \times CG + \overline{m - 72} \times BH - \overline{m' - 240} \times AI + \overline{m' - 600} \times K$, the Equation muſt have ſome Roots imaginary.

For an Example, If the four Roots of the Biquadratick Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ are real Quantities, it will follow equally from the vth, viith, ixth, and xith Propoſitions, that $\frac{3}{8}A^2$ muſt be greater than B, and that $\frac{3}{8}C^2$ muſt exceed BD. The viith further ſhews that $\frac{5}{12}B^2$ muſt exceed $AC - D$; the ixth demonſtrates that $\frac{4}{9}B^2$ muſt exceed AC; but our Rule deduced from Prop xi. ſhews that $2B^2$ muſt exceed $5AC - 8D$, the exceſs being $\frac{1}{2}a'$, and the Rule deduced from the ſecond Theorem of the xiith Propoſition ſhews that B^2 muſt always exceed $2AC + 4D$, the Exceſs being $\frac{1}{4}a''$. It appears from ſeveral preceeding Propoſitions, that if the Roots of the Equation have all the ſame Sign, then AC muſt exceed $16D$: Let the Exceſſes $5B^2 - 12AC + 12D = p$, $4B^2 - 9AC = q$, $AC - 16D = s$; and it is plain that a' $(= 4B^2 - 10AC + 16D) = q - s$

$-s = \frac{2}{5} \times \overline{2p - s}$; and that $a'' = q + s = \frac{2}{5} \times \overline{2p + 4s}$. Let us ſuppoſe,

1. That s is poſitive, then it is manifeſt that if either p or q be negative, a' muſt alſo be found negative, and conſequently that when the viith or ixth Propoſitions ſhew any Roots to be imaginary, the xith Propoſition muſt diſcover them at the ſame time. But as a' $(= q - s = \frac{2}{5} \times \overline{2p - s})$ may be found negative when p and q are both poſitive, it follows that the Rule we have deduced from the xith Propoſition may diſcover imaginary Roots in an Equation, that do not appear by the preceeding Propoſitions: Thus if you examine the Equation $x^4 - 6x^3 + 10x^2 - 7x + 1$ by Sir *Iſaac Newton*'s Rule, or by our viith Propoſition, no imaginary Roots appear in it from either. But ſince $2B^2 - 5AC + 8D \left(= \frac{1}{2}a'\right) = 200 - 210 + 8 = -2$ is in this Equation negative, it is manifeſt that two Roots of the Equation muſt be imaginary. Let us ſuppoſe

2. That s is negative, and that from the Signs of the Terms of the Equation, it appears that ſome Roots are poſitive and ſome negative; then in Order to ſee if the Equation has any imaginary Roots, the moſt uſeful Rule is that we deduced from the ſecond Theorem of Prop. xii. *viz.* that if B^2 does not exceed $2AC + 4D$ ſome of the Roots of the Equation muſt be imaginary: For the Exceſs of B^2 above $2AC + 4D$ being $\frac{1}{4}a'' = \frac{1}{4} \times \overline{q + s} = \frac{1}{10} \times \overline{2p + 4s}$, and s

being negative, it is manifeſt, that if q or p be negative $\frac{1}{4}a''$ muſt be negative; and that $\frac{1}{4}a''$ may be negative when q and p are both poſitive; that is, This Rule muſt always diſcover ſome Roots to be imaginary when the viith or ixth Propoſitions diſcover any impoſſible Roots in an Equation; and will very often diſcover ſuch Roots in an Equation when theſe Propoſitions diſcover none. For Example, if you examine the Equation $x^4 + 5x^3 + 6x^2 - x - 12 = 0$, you will diſcover no imaginary Roots in it by the viith or ixth Propoſitions; and though $AC - 16D \; (= s)$ be negative, it does not follow, that the Equation has any impoſſible Roots, becauſe it appears from the Signs of the Terms, that the Equation has Roots affected with different Signs. But ſince $B^2 - 2AC - 4D \; (= 36 + 10 - 48 = -2)$ is negative, it appears from our Rule, that the Equation muſt have ſome imaginary Roots.

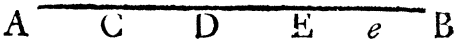
I might ſhew in the next Place, how the Rules deduced from the xith and xiith Propoſitions may be extended ſo as to diſcover when more than two Roots of an Equation are imaginary, and in general to determine the Number of imaginary Roots in any Equation; but as it would require a long Diſcuſſion, and ſome *Lemmata* to demonſtrate this ſtrictly, I ſhall only obſerve that theſe xith and xiith Propoſitions will be found to be ſtill the moſt uſeful of all thoſe we have given for that Purpoſe. To give one Example of this; If we are to examine the Equation $\underset{+}{x^4} \underset{-}{-\,4ax^3} \underset{+}{+\,6a^2x^2} \underset{-}{-\,4ab^2x} \underset{+}{+\,b^4} = 0$ by Sir *Iſaac Newton*'s Rule, it is found to have four impoſſible Roots when a is greater than b; for though the Square of the ſecond Term multiplied by $\frac{3}{8}$ be equal to the Product of the firſt and third Terms, yet in that Caſe, in applying Sir *Iſaac Newton's* Rule, the Sign — ought to be placed under the ſecond Term, and the ſame is to be ſaid of the Square of the fourth Term. The Rule deduced from the viith Propoſition ſhews four Roots imaginary, when a is greater than b, and alſo when b^2 is greater than 15 a^2; but a Rule founded on the xith Propoſition, ſhews the four Roots to be imaginary always when a exceeds b, or when b^2 exceeds 9 a^2; from which the Excellency of this Rule above theſe two is manifeſt. I have ſaid ſo much of Biquadratick Equations, that I muſt leave it to thoſe that are willing to take the Trouble, to make like Remarks on the higher Sorts of Equations.

In inveſtigating the preceeding Propoſitions, when I found my ſelf obliged to go through ſo intricate Calculations, I often attempted to find ſome more eaſy Way of treating this Subject. The following was of conſiderable Uſe to me,and may perhaps be entertaining to you. By it, I inveſtigate ſome *maxima* in a very eaſy Manner, that could not be demonſtrated in the common Way with ſo little Trouble.

LEMMA V. Let the given Line AB be divided any where in P and the Rectangle of the Parts AP and PB will be a *maximum* when theſe Parts are equal.

This is manifeſt from the Elements of *Euclid*.

LEMMA VI. If the Line AB is divided into any Number of Parts AB, CD, DE, EB, the Product of all thoſe Parts multiplied into one another will be a *maximum* when the Parts are equal amongſt themſelves. For let the Point D be where you will, it is manifeſt that if DB be biſſected in E, the Product AC × CD × DE × EB will be greater than AC × CD × De × eB



becauſe by the laſt Lemma DE × EB is greater than De × eB; and for the ſame reaſon AD and CE muſt be biſſected in C and D; and conſequently all the Parts AC, CD, DE, EB muſt be equal amongſt themſelves, that their Product may be a *maximum.*

Lemma VII. The Sum of the Products that can be made by multiplying any two Parts of AB by one another is a *maximum* when the Parts are equal. The Sum of theſe Products is AC × CB + CD × DB + DE × EB: Now that DE × EB may be a *maximum*, DB muſt be biſſected in E by the vth Lemma, and for the ſame reaſon AD and CE muſt be biſſected in C and D, that is all the Parts, AC, CD, DE, EB muſt be equal, that the Sum of all theſe Products may be a *maximum.*

Lemma VIII. The Sum of the Products of any three Parts of the Line AB is a *maximum*, when all the Parts are equal. For that Sum is $AC \times CD \times DE + EB \times \overline{AC \times CD + AC \times DE + CD \times DE}$; and ſuppoſing the Point E given, it is manifeſt that AE muſt be equally triſected in C and D that AC × CD × DE may be a *maximum* by Lemma vi. and that AC × CD + AC × DE + CD × DE may be a *maximum* by Lemma viith. From which it is manifeſt that all the Parts AC, CD, DE, EB muſt be equal, that the Sum of the Products of any three of them may be a *maximum.*

Lemma IX. It is manifeſt that this way of reaſoning is general, and that the Sum of any Quantities being given, the Sum of all the Products that can be made by multiplying any given Number of them by one another, muſt be a *maximum* when theſe Quantities are equal. But the Sum of the Squares, or of any pure Powers of theſe Quantities, is a *minimum*, when the Quantities are equal.

THEOREM.

Suppoſe $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5}$ &c. $= 0$, *to be an Equation that has not all its Roots equal to one another: Let* r *expreſs the Dimenſions of any Coefficient* D, *and let* $l = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ &c. *taking as many Factors as there are Units in* r; *then ſhall* $\frac{l}{n^r} \times A^r$ *be always greater than* D, *if the Roots of the Equation are real Quantities affected with the ſame Sign.*

This may be demonſtrated from the preceeding Propoſitions: But to demonſtrate it from the laſt Lemmata, let us aſſume an Equation that has all its Roots equal to one another, and the Sum of all its Roots equal to A, the Sum of the Roots of the propoſed Equation. This Equation will be $\overline{x - \frac{1}{n}A}\Big|^{n} = 0$, or $x^n - Ax^{n-1} + n \times \frac{n-1}{2} \times \frac{A^2}{n^2} x^{n-2} - n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{A^3}{n^3} x^{n-3}$ &c. $= 0$ and if r expreſs the Dimenſions of the Coefficient of any Term of this Equation (or the Number of Terms which preceed it) it is manifeſt that the Term it ſelf will be $l \times \frac{A^r}{n^r} x^{n-r}$: But by the Suppoſition $D x^{n-r}$ is the Correſponding Term in the propoſed Equation, and D muſt be the Sum of all the Products that can be made by multiplying as many Roots of that Equation by one another, as there are Units in r; and $\frac{l A^r}{n^r}$ muſt be the Sum of the like Products of the Roots of the other Equation; which muſt be the greater Quantity by the preceeding Lemmata, becauſe its Roots are equal amongſt themſelves, and their Sum is equal to the Sum of the Roots of the propoſed Equation; and the Sum of ſuch Products is a *maximum* when the Roots are equal amongſt themſelves. By purſuing this Method, it may be demonſtrated that $\left.\overline{\frac{2B}{n \times \overline{n-1}}}\right|^{\frac{r}{2}} \times l$ muſt always exceed the Coefficient prefixed to the Term x^{n-r} in an Equation whoſe Roots are all real Quantities affected with the ſame Sign; providing that r be a Number greater than 2; and alſo that $\left.\overline{\frac{2 \times 3C}{n \times \overline{n-1} \times \overline{n-2}}}\right|^{\frac{r}{3}} \times l$ muſt exceed the ſame Coefficient, if r be any Number greater than 3.

It is eaſy to continue theſe Theorems.

The third Method which I mentioned in the Beginning of this Letter, is deduced from the Conſideration of the Limits of the Roots of Equations; and though it is explained by ſome Authors already, yet as I demonſtrate and apply it to this Subject in a different Manner, I ſhall add a ſhort Account of it.

LEMMA X. If you transform the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ into one that ſhall have each of its Roots leſs than the reſpective Values of x by a given Difference e; ſuppoſe $y = x - e$ or $x = e + y$ and the transformed Equation, the Order of the Terms being inverted, will have this Form.

$$\begin{array}{lllll} e^4 & + 4e^3y & + 6e^2y^2 & + 4ey^3 & + y^4 = 0. \\ - Ae^3 & - 3Ae^2y & - 3Aey^2 & - Ay^3 & \\ + Be^2 & + 2Bey & + By^2 & & \\ - Ce & - Cy & & & \\ + D & & & & \end{array}$$

Where it is manifeſt,

1. That the firſt Term $e^4 - Ae^3 + Be^2 - Ce + D$ is the Quantity that ariſes by ſubſtituting e in Place of x in the propoſed Equation $x^4 - Ax^3 + Bx^2 - Cx + D$.

2. That the Coefficient of the ſecond Term $4e^3 - 3Ae^2 + 2Be - C$ is the Quantity that ariſes by multiplying each Part of the firſt $e^4 - Ae^3 + Be^2 - Ce + D$ by the Index of e in that Part, and dividing the Product by e.

3. That the Coefficient of the third Term $6e^2 - 3Ae + B$ is the Quantity that ariſes from the preceeding Coefficient $4e^3 - 3Ae^2 + 2Be - C$ by multiplying each Part by the Index of e in it, and dividing the Product by $2e$.

4. That the Coefficient of the fourth Term ariſes in like Manner from the preceeding, only you now divide by $3e$; and in general, the Coefficient of any Term may be deduced from the Coefficient of that Term which preceeds it, by multiplying each Part of

the preceeding Coefficient by the Index of e in that Part, and dividing the Product by e and by the Index of y, in the Term whoſe Coefficient is required.

LEMMA XI. If any Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$ be transformed in the ſame Manner, by ſuppoſing $x = y - e$ or $x = e + y$, and conſequently $x^n = \overline{e+y}|^{n}$, $Ax^{n-1} = A\times\overline{e+y}|^{n-1}$, $Bx^{n-2} = B\times\overline{e+y}|^{n-2}$ &c. The transformed Equation will have this Form, the Order of the Terms being inverted,

$$\begin{array}{llll}
e^n + ne^{n-1}y + n\times\frac{n-1}{2}\times e^{n-2}y^2 \text{ \&c.} = 0 \\
-Ae^{n-1} - \overline{n-1}\times Ae^{n-2}y & - \overline{n-1}\times\frac{n-2}{2}\times Ae^{n-3}y^2 \text{ \&c.} \\
+Be^{n-2} + \overline{n-2}\times Be^{n-3}y & + \overline{n-2}\times\frac{n-3}{2}\times Be^{n-4}y^2 \text{ \&c.} \\
-Ce^{n-3} - \overline{n-3}\times Ce^{n-4}y & - \overline{n-3}\times\frac{n-4}{2}\times Ce^{n-5}y^2 \text{ \&c.} \\
\text{\&c.} \qquad\qquad \text{\&c.} & \qquad \text{\&c.}
\end{array}$$

Where it is manifeſt,

1. That the firſt Term $e^n - Ae^{n-1} + Be^{n-2} - Ce^{n-3}$ &c. is the Quantity that ariſes by ſubſtituting e in the Place of x in the propoſed Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c.

2. That the Coefficient of the ſecond Term $ne^{n-1} - \overline{n-1}\times Ae^{n-2} + \overline{n-2}\times Be^{n-3} - \overline{n-3}\times Ce^{n-4}$ &c. is deduced from the preceeding $e^n - Ae^{n-1} + Be^{n-2} - Ce^{n-3}$ &c. by multiplying each of its Parts by the Index of e in that Part, and dividing by e.

3. That the Coefficient of the third Term is deduced from the Coefficient of the ſecond Term, by multiplying after the ſame manner, each of its Parts by the Index of e and dividing by $2e$. In general, the Coefficient of any Term y^r is deduced from the Coefficient of the preceeding Term, that is of y^{r-1} by multiplying every Part of that Coefficient by the Index of e in it, and dividing the Product by re.

LEMMA XII. If you ſubſtitute any two Quantities K and L in the Place of x in $x^4 - Ax^3 + Bx^2 - Cx + D$, and the Quantities that reſult from theſe Subſtitutions be affected with contrary Signs, the Quantities K and L muſt be *Limits* of one or more real Roots of the Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$. That is, one of theſe Quantities muſt be greater, and the other leſs than one or more Roots of that Equation.

For if you ſuppoſe that $a, b, c, d,$ are the Roots of that Equation, then it is plain from the *Geneſis* of Equations, that $x^4 - Ax^3 + Bx^2 - Cx + D = \overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$; and therefore K and L being ſubſtituted for x in $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, the Product becomes in the one Caſe poſitive, and in the other negative; ſo that one of the Factors $x-a$, $x-b$, $x-c$, $x-d$ muſt have a Sign when K is ſubſtituted for x in it, contrary to the Sign which it is affected with when L is ſubſtituted in in it for x, ſuppoſe that Factor to be $x-b$; and ſince $K-b$ and $L-b$ are Quantities whereof the one is poſitive, and the other negative, it is manifeſt that b one of the Roots of the Equation muſt be leſs than one, and greater than the other of the two Quantities K and L: So that K and L muſt be the *Limits* of the Root b.

I ſay further, that the Root whereof K and L are *Limits*, muſt be a real Root of the Equation; for the Product of the Factors that involve impoſſible Roots in an Equation can never have its Signs changed by ſubſtituting any real Quantity whatſoever in place of x; becauſe the Number of ſuch Roots is always an even Number, and the Product of any two of theſe Roots ſuch as $x - m - \sqrt{-n}$, and $x - m + \sqrt{-n}$ is $\overline{x - m}|^2 + n^2$ which muſt be always poſitive, whatever Quantity be ſubſtituted for x while n remains poſitive, that is, while theſe two Roots are impoſſible.

LEMMA XIII. If you ſubſtitute K and L for x in $x^n - Ax^{n-1} + Bx^{n-2}$ &c. and the Quantities that reſult be affected with contrary Signs, then ſhall K and L be the *Limits* of one or more real Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$. This may be demonſtrated after the ſame Manner as the laſt Lemma.

THEOREM I. If a, b, c, d are the Roots of the Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, they ſhall be the *Limits* of the Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$.

Suppoſe a to be the leaſt Root of the biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, b the ſecond Root, c the third, and d the fourth, and the Values of y in the Equation in the xth Lemma, will be $a - e$, $b - e$, $c - e$, $d - e$; then by ſubſtituting ſucceſſively a, b, c, d for e in that Equation of y, one of the Values of y will vaniſh in every Subſtitution, and the firſt Term of the Equation of y, *viz.* $e^4 - Ae^3 + Be^2 - Ce + D$ vaniſhing, the Equation will be reduced to a Cubick of this Form.

$$\begin{array}{llll} & 4e^3 & + \;6e^2y + 4ey^2 + y^3 = 0 \\ -\; 3Ae^2 & -\; 3Aey & -\; Ay^2 & \\ +\; 2Be & +\; By & & \\ -\; C & & & \end{array}$$

And conſequently $4e^3 - 3Ae^2 + 2Be - C$ muſt be the Product of the three remaining Values of y having its Sign changed; that is, it muſt be equal to $-\overline{b-a}\times\overline{c-a}\times\overline{d-a}$ when e is ſuppoſed equal to a, it muſt be $-\overline{a-b}\times\overline{c-b}\times\overline{d-b}$ when $e=b$; it muſt be $-\overline{a-c}\times\overline{b-c}\times\overline{d-c}$ when $e=c$; and it muſt be $-\overline{a-d}\times\overline{b-d}\times\overline{c-d}$ when $c=d$. Now it is manifeſt that theſe Products $\overline{b-a}\times\overline{c-a}\times\overline{d-a}$, $\overline{a-b}\times\overline{c-b}\times\overline{d-b}$, $\overline{a-c}\times\overline{b-c}\times\overline{d-c}$, $\overline{a-d}\times\overline{b-d}\times\overline{c-d}$ muſt be affected with the Signs $+$, $-$, $+$, $-$ reſpectively; the firſt being the Product of three poſitive Quantities, the ſecond the Product of one negative and two poſitives, the third the Product of two negatives and one poſitive, and the fourth the Product of three negatives. Therefore ſince by ſubſtituting a, b, c, d for e in the Quantity $4e^3 - 3Ae^2 + 2Be - C$, it becomes alternately a poſitive and a negative Quantity, it follows from the laſt Lemma that a, b, c, d muſt be the *Limits* of the Roots of the Equation $4e^3 - 3Ae^2 + 2Be - C = 0$, or of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$.

COR. It follows from this Theorem, that if $a'\,b'$ and c' are the three Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$, they muſt be *Limits* betwixt a, b, c, d the Roots of the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$: For if a, b, c, d are *Limits* of the

Roots a', b', and c'; theſe Roots converſely muſt be *Limits* betwixt a, b, c and d.

Theorem II. Multiply the Terms of any Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ by any Arithmetical Series of Quantities $l + 4m$, $l + 3m$, $l + 2m$, $l + m$, l, and the Roots of the Biquadratick a, b, c, d will be the *Limits* of the Roots of the Equation that reſults from that Multiplication that is of the Equation.

$$\begin{array}{l} lx^4 - lAx^3 + lBx^2 - lCx + lD = 0 \\ +4mx^4 - 3mAx^3 + 2mBx^2 - mCx \end{array}$$

Suppoſe that ſubſtituting the Roots a, b, c, d of the biquadratick Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ ſucceſſively, for x in $4x^3 - 3Ax^2 + 2Bx - C$, the Quantities that reſult are $-R$, $+S$, $-T$, $+Z$; while $x^4 - Ax^3 + Bx^2 - Cx + D$ is in every Subſtitution equal to nothing; and it is manifeſt that the Quantity

$$\begin{array}{l} +lx^4 - lAx^3 + lBx^2 - lCx + lD \\ +4mx^4 - 3mAx^3 + 2mBx^2 - mCx \end{array}$$

will become (when a, b, c, d are ſubſtituted ſucceſſively in it for x) equal to $-mRx$, $+mSx$, $-mTx$, $+mZx$; where the Signs of theſe Quantities being alternately negative and poſitive, it follows that a, b, c, d muſt be *Limits* of that Equation by Lemma xii.

Cor. Hence it follows, that a, b, c and d are *Limits* of the Roots of the Cubick Equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$, and converſely, that the Roots of this Cubick are *Limits* of the Roots of the biquadratick Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, for multiplying the Terms of this biquadratick Equation by the Arithmetical Progreſſion $0, -1, -2, -3, -4$, the Cubick $Ax^3 - 2Bx^2 + 3Cx - AD = 0$ ariſes.

THEOREM III. *In general, the Roots of the Equation* $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$, *are the* Limits *of the Roots of the Equation* $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$, *or of any Equation that is deduced from it by multiplying its Terms by any Arithmetical Progreſſion* $l \mp d, l \mp 2d, l \mp 3d$ &c. *and converſely the Roots of this new Equation will be the* Limits *of the Roots of the propoſed Equation* $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$.

This Theorem is demonſtrated from the xith and xiiith Lemmata in the ſame manner as the preceeding Theorems were demonſtrated from the xth and xiith. From theſe Theorems it is eaſy to infer all that is delivered by the Writers of Algebra on this Subject.

THEOREM IV. *The Equation* $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$ *will have as many imaginary Roots as the Equation* $nx^{n-1} - \overline{n-1} \times Ax^{n-2} - \overline{n-2} \times Bx^{n-3}$ &c. $= 0$, *or the Equation* $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3}$ &c. $= 0$.

Suppoſe that any Root of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$, as p becomes imaginary; and the two Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c $= 0$, which by Theorem III. ought to be its *Limits*, cannot both be real Quantities; for it is manifeſt from the Demonſtration of Theorem I. that if they are real Quantities, then being ſubſtituted for x in $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. the Quantities that reſult muſt have contrary Signs, and conſequently the Root p, whereof they are *Limits*, muſt be a real Root;

which is againſt the Suppoſition. The ſame is true of the Equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3}$ &c. $= 0$, for the ſame Reaſon.

Cor. The biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, will have two imaginary Roots, if two Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ be imaginary; or if two Roots of the Equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ be imaginary. But two Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ muſt be imaginary, when two Roots of the Quadratick $6x^2 - 3Ax + B = 0$, or of the Quadratick $3Ax^2 - 4Bx + 3C = 0$ are imaginary, becauſe the Roots of theſe quadratick Equations are the *Limits* of the Roots of that Cubick, by the third Theorem; and for the ſame reaſon two Roots of the Cubick Equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ muſt be imaginary, when the Roots of the quadratick $3Ax^2 - 4Bx + 3C = 0$, or of the quadratick $Bx^2 - 3Cx + 6D = 0$ are impoſſible. Therefore two Roots of the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ muſt be imaginary when the Roots of any one of theſe three quadratick Equations $6x^2 - 3Ax + B = 0$, $3Ax^2 - 4Bx + 3C = 0$, $Bx^2 - 3Cx + 6D = 0$ become imaginary; that is, when $\frac{3}{8}A^2$ is leſs than B, $\frac{4}{9}B^2$ leſs than AC, or $\frac{3}{8}C^2$ leſs than BD.

Cor. II. By proceeding in the ſame manner, you may deduce from any Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$, as many quadratick Equations as there are Terms excepting the firſt and laſt whoſe Roots muſt be all real Quantities, if the propoſed Equation has no imaginary Roots. The Quadratick deduced from the three firſt Terms $x^n - Ax^{n-1} + Bx^{n-2}$ will manifeſtly have this Form, $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}\ \&c. \times x^2 - \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}\ \&c. \times Ax + \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5}\ \&c. \times B = 0$, continuing the Factors in each till you have as many as there are Units in $n-2$. Then dividing the Equation by all the Factors $n-2$, $n-3$ &c. which are found in each Coefficient, the Equation will become $n \times \overline{n-1} \times x^2 - \overline{n-1} \times 2Ax + 2 \times 1 \times B = 0$, whoſe Roots will be imaginary by Prop. i. when $n \times \overline{n-1} \times 2 \times 4B$ exceeds $\overline{n-1}|^2 \times 4A^2$, or when B exceeds $\frac{n-1}{2n}A^2$, ſo that the propoſed Equation muſt have ſome imaginary Roots when B exceeds $\frac{n-1}{2n}A^2$; as we demonſtrated after another Manner in the vth Propoſition. The Quadratick Equation deduced in the ſame Manner from the three firſt Terms of the Equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3}\ \&c. = 0$, will have this Form $\overline{n-1} \times \overline{n-2} \times \overline{n-3}\ \&c. \times Ax^2 - \overline{n-2} \times \overline{n-3} \times \overline{n-4}\ \&c. \times 2Bx + \overline{n-3} \times \overline{n-4} \times \overline{n-5}\ \&c. \times 3C = 0$; which by dividing by the Factors common to all the Terms, is reduced to $\overline{n-1} \times \overline{n-2} \times Ax^2 - \overline{n-2} \times 4Bx + 6C = 0$, whoſe Roots muſt be imaginary when $\frac{2}{3} \times \frac{n-2}{n-1} \times B^2$ is leſs than AC; and therefore in that caſe ſome Roots of the propoſed Equation muſt be imaginary.

COR. III. In general, let $Dx^{n-r+1} - Ex^{n-r} + Fx^{n-r-1}$ be any three Terms of the Equation, $x^n -$

$Ax^{n-1}+Bx^{n-2}$ &c. $=0$, that immediately follow one another, multiply the Terms of this Equation firſt by the Progreſſion $n, n-1, n-2$, &c. then by the Progreſſion $n-1, n-2, n-3$, &c. then by $n-2, n-3, n-4$, &c. till you have multiplied by as many Progreſſions as there are Units in $n-r-1$: Then multiply the Terms of the Equation that ariſes, as often by the Progreſſion $0, 1, 2, 3$ &c. as there are Units in $r-1$, and you will at length arrive at a Quadratick of this Form,

$$\overline{n-r+1}\times\overline{n-r}\times\overline{n-r-1}\times\overline{n-r-2}\text{ \&c. }\times\overline{r-1}$$
$$\times\overline{r-2}\times\overline{r-3}\times\overline{r-4}\text{ \&c. }Dx^2$$
$$-\overline{n-r}\times\overline{n-r-1}\times\overline{n-r-2}\times\overline{n-r-3}\text{ \&c.}$$
$$\times r\times\overline{r-1}\times\overline{r-2}\times\overline{r-3}\text{ \&c.}\times Ex$$
$$+\overline{n-r-1}\times\overline{n-r-2}\times\overline{n-r-3}\times\overline{n-r-4}\text{ \&c.}$$
$$\times\overline{r+1}\times r\times\overline{r-1}\times\overline{r-2}\text{ \&c.}\times F=0,$$

and dividing by the Factors $n-r-1, n-r-2$, &c. and $r-1, r-2$ &c. which are found in each Coefficient, this Equation will be reduced to $\overline{n-r+1}\times\overline{n-r}\times2\times1\times Dx^2-\overline{n-r}\times2\times r\times2Ex+2\times1\times\overline{r+1}\times rF=0$, whoſe Roots muſt be imaginary (by Prop i.) when $\frac{n-r}{n-r+1}\times\frac{r}{r+1}\times E^2$ is leſs than DF. From which it is manifeſt that if you divide each Term of this Series of Fractions $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \frac{n-3}{4}$, &c. $\frac{n-r+1}{r}, \frac{n-r}{r+1}$ by that which preceeds it, and place the Quotients above the Terms of the Equation $x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}$ &c. $=$

&c. $= 0$, beginning with the ſecond: Then if the Square of any Term multiplied by the Fraction over it be found leſs than the Product of the adjacent Terms, ſome of the Roots of that Equation muſt be imaginary Quantities. There remain many things that might be added on this Subject, but I am afraid you will think I have ſaid as much of it as it deſerves; and therefore I ſhall only add the Demonſtration of ſome Algebraick Rules and Theorems that are very eaſily deduced from the xi^th^ Lemma.

I. The Rule for diſcovering when two or more Roots of an Equation are equal, immediately follows from that Lemma, Suppoſe that two Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$ are equal, and two Values of y (which is equal always to $x-e$) will be equal. Suppoſe that e is equal to one of thoſe two equal Values of x; and two Values of y will vaniſh, and conſequently y^2 muſt enter each of the Terms of the Equation of y; and therefore in this Caſe the firſt and ſecond Term of the Equation of y in Lemma xi^th^ muſt vaniſh, that is, $e^n - Ae^{n-1} + Be^{n-2} - Ce^{n-3}$ &c. $= 0$ and $ne^{n-1} - \overline{n-1} \times Ae^{n-2} + \overline{n-2} \times Be^{n-3} - \overline{n-3} \times Ce^{n-4}$ &c. $= 0$ at the ſame time; and conſequently theſe two Equations muſt have one Root common, which muſt be one of thoſe Values of x that were ſuppoſed equal to each other. It is manifeſt therefore that when two Values of x are equal in the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$, one of them muſt be a Root of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$.

If three Values of x be ſuppoſed equal amongſt themſelves and to e, then three Values of y $(= x - e)$ will vaniſh, and the firſt three Terms of the Equation of y

in Lemma xi. will vaniſh, and therefore $n \times \overline{n-1} \times e^{n-2} - \overline{n-1} \times \overline{n-2} \times Ae^{n-3} + \overline{n-2} \times \overline{n-3} \times Be^{n-4}$ &c $= 0$; and one of the equal Values of x will be a Root of this laſt Equation, and two of them will be Roots of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$. In general, it appears that if the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$ have as many Roots equal amongſt themſelves as there are Units in S, then ſhall as many of thoſe be Roots of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$ as there are Units in $S - 1$; as many of them ſhall be Roots of the Equation $n \times \overline{n-1} \times x^{n-2} - \overline{n-1} \times \overline{n-2} \times Ax^{n-3} + \overline{n-2} \times \overline{n-3} \times Bx^{n-4}$ &c. $= 0$, as there are Units in $S - 2$; and ſo on.

II. The general Rule which Sir *Iſaac Newton* has given in the *Article de limitibus Equationum* for finding a *Limit* greater than any of the Values of x immediately follows from the xi^th^ Lemma; for it is manifeſt that if e be ſuch a Quantity as ſubſtituted in all the Coefficients of the Equation of y, *viz.* in $e^n - Ae^{n-1} + Be^{n-2}$ &c. $ne^{n-1} - \overline{n-1} \times Ae^{n-2} + \overline{n-2} \times Be^{n-3}$ &c. $n \times \frac{n-1}{2} \times e^{n-2} - \overline{n-1} \times \frac{n-2}{2} \times Ae^{n-3} + \overline{n-2} \times \frac{n-3}{2} \times Be^{n-4}$ &c. gives the Quantities that reſult all poſitive; then there being no Changes of the Signs of the Equation of y in this caſe, all its Values muſt be negative; and ſince y is always equal to $x - e$ it follows that e muſt be a greater Quantity than any of the Values of x; that is, it muſt be a

Limit greater than any of the Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$.

III. From this xith Lemma ſome important Theorems in the Method of *Series*, and of *Fluxions*, and the Reſolution of Equations are demonſtrated with great Facility; it is obvious that the Coefficient of the ſecond Term of the Equation of y in that Lemma is the *Fluxion* of the firſt Term divided by the *Fluxion* of e; the Coefficient of the third Term is the ſecond *Fluxion* of that firſt Term divided by $2\dot{e}^2$, ſuppoſing e to flow uniformly. The third Term is the third *Fluxion* of the firſt Term divided by $2 \times 3\,\dot{e}^3$; and ſo on. Therefore ſuppoſing $e^n - Ae^{n-1} + Be^{n-2}$ &c. $= c$, the Equation for determining y will be $c + \frac{\dot{c}}{\dot{e}}y + \frac{\ddot{c}}{1 \times 2\,\dot{e}^2}y^2 + \frac{\dddot{c}}{1 \times 2 \times 3\,\dot{e}^3}y^3$ &c. $= 0$; and hence, when e is near the true Value of x, Theorems may be deduced for approximating to y, and conſequently to x, which is ſuppoſed equal to $y + e$.

IV. Let AP ($= x$) be the Abſciſs and PM ($= z$) the Ordinate of any Curve BLM; and ſuppoſe any other Abſciſs AK $= e$ and Ordinate KL $= c$, then ſhall z ($=$ PM) $= c \mp \frac{\dot{c}}{\dot{e}}y + \frac{\ddot{c}}{2\dot{e}^2}y^2 \mp \frac{\dddot{c}}{2 \times 3\,\dot{e}^3}y^3 + \frac{\ddddot{c}}{2 \times 3 \times 4\,\dot{e}^4}y^4$ &c.

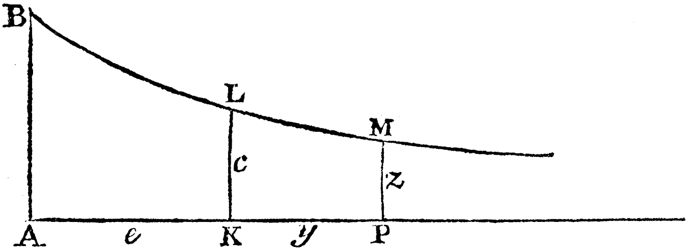
For let z be ſuppoſed equal to any Series conſiſting of given Quantities, and the Powers of x, as to $Ax^n + Bx^r + Cx^s$ &c. and ſubſtituting $e \mp y$ for x, we ſhall find after the manner of the xith Lemma,

$$z = Ae^{n} \mp nAe^{\overline{n-1}}y + n \times \frac{n-1}{2} \times Ae^{\overline{n-2}}y^{2} \text{ \&c.}$$

$$+ Be^{r} \mp rBe^{r-1}y + r \times \frac{r-1}{2} \times Be^{r-2} \times y^{2} \text{\&c.}$$

$$+ Ce^{s} \mp sCe^{s-1}y + s \times \frac{s-1}{2} \times Ce^{s-2}\,y^{2} \text{ \&c.}$$

&c. &c. &c.



But when $x = e$ then $z = c = Ae^{n} + Be^{r} + Ce^{s}$ &c. $\dot{c} = nAe^{n-1}\dot{e} + rBe^{r-1}\dot{e} + sCe^{s-1}\dot{e}$ &c. $\ddot{c} = n \times \overline{n-1} \times Ae^{n-2}\dot{e}^{2} + r \times \overline{r-1} \times Be^{r-2}\dot{e}^{2} + s \times \overline{s-1} \times Ce^{s-2}\dot{e}^{2}$ &c. and therefore $z = c \mp \frac{\dot{c}}{\dot{e}}y + \frac{\ddot{c}}{2\dot{e}^{2}}y^{2} \mp \frac{\dddot{c}}{2 \times 3\dot{e}^{3}}y^{3}$ &c. After the ſame manner you will find that $c = z \pm \frac{\dot{z}}{\dot{x}}y + \frac{\ddot{z}}{2\dot{x}^{2}}y^{2} \pm \frac{\dddot{z}}{2 \times 3\dot{x}^{3}}y^{3}$ &c. for $c = Ae^{n} + Be^{r} + Ce^{s}$ &c. $= A \times \overline{x \pm y}|^{n} + B \times \overline{x \pm y}|^{r} + C \times \overline{x \pm y}|^{s}$ &c. $= z \pm \frac{\dot{z}}{\dot{x}}y$

$+\frac{\ddot{z}}{2\dot{x}^{2}}y^{2}$ &c. The Area KLMP is equal to the Fluent of $z\dot{y}$ or of $c\dot{y}$, but

$$cy = z\dot{y} \pm \frac{\dot{z}}{x}y\dot{y} + \frac{\ddot{z}}{2x^{2}}y^{2}\dot{y} \pm \frac{\dddot{z}}{2\times 3\,\dot{x}^{2}}y^{3}\dot{y}\ \&c.$$

and $$z\dot{y} = c\dot{y} \mp \frac{\dot{c}}{\dot{e}}y\dot{y} + \frac{\ddot{c}}{2\dot{e}^{2}}y^{2}\dot{y} \mp \frac{\dddot{c}}{2\times 3\,\dot{e}^{2}}y^{3}\dot{y}\ \&c.$$

And conſequently by finding the Fluents

$$\text{KLMP} = cy \mp \frac{\dot{c}}{2\dot{e}}y^{2} + \frac{\ddot{c}}{2\times 3\,\dot{e}^{2}}y^{3} \mp \frac{\dddot{c}}{2\times 3\times 4\dot{e}^{3}}y^{4}\ \&c.$$

or $$\text{KLMP} = zy \pm \frac{\dot{z}}{2\dot{x}}y^{2} + \frac{\ddot{z}}{2\times 3\,\dot{x}^{2}}y^{3} \pm \frac{\dddot{z}}{2\times 3\times 4\,\dot{x}^{3}}y^{4}\ \&c.$$

This laſt is the Theorem publiſhed by the learned Mr. *Bernouilli* in the *Acta Lipſiæ* 1694. It is now high Time to conclude this long Letter; I beg you may accept of it as a Proof of that Reſpect and Eſteem with which

I am,

S I R,

Your moſt Obedient,

Moſt Humble Servant,

Colin Mac Laurin.